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PHASE-ONLY NULLING AS A NONLINEAR PROGRAMMING PROBLEM

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The problem of calculating the minimum phase-only weight perturbations required to impose nulls in the pattern of a linear antenna array is nonlinear and cannot be solved analytically. The problem is, however, an example of nonlinear programming problem and can be solved numerically. This report describes the performance of two nonlinear programming computer codes, LPNLP and VMCON, on the minimized weight perturbation phase-only null synthesis problem. Both codes are effective in general, although convergence problems may be encountered if nulls are required to be imposed at closely

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spaced locations in high sidelobe regions of the pattern.

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1. INTRODUCTION

Current interest in the subject of phase-only control[1-7] of array weights for adaptive cancellation of interference and for array antenna pattern null synthesis[8-14] is the result of the widespread use of phased arrays. The restriction of the perturbations of the array weights to be of the phases only results in complexities in null synthesis not present when the array weights can be freely perturbed in both phase and amplitude. The complexities stem from the fact that the equations for imposing pattern nulls with phase-only control are nonlinear, whereas the null equations for combined phase and amplitude control are linear. The nonlinear equations cannot be solved exactly analytically and hence alternate methods must be found to obtain solutions.

In null synthesis it is typically the case that in addition to seeking a pattern with nulls at a set of specified locations, it is also desired that the synthesized patterns should satisfy criteria such as sidelobes not exceeding a given level or sidelobes having a specified envelope. One way of coming close to achieving these goals is to start with a set of array weights that correspond to a pattern which is acceptable in all respects except that of possessing nulls at all the desired locations, and to then

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(Due to the large number of references cited above, they will not be listed here. See References, page 17.)

perturb the array weights as little as possible so as to impose pattern nulls at the set of specified locations. It can then be anticipated that the resulting pattern change will be minimized and that the perturbed pattern will bear a close resemblance to the original pattern.

In previous reports we have investigated various aspects of the minimized weight perturbation null synthesis problem when the weight perturbations are restricted to be of the phases only. In Reference 11 we showed that when the phase perturbations are small, the null constraint equations can be linearized and a solution found to the problem of imposing nulls subject to minimized weight perturbations in a completely similar way to the case of combined phase and amplitude perturbations. The quality of the nulls obtained via this linearization suffers considerably in general, however, compared with the null depths achieved with the exact solutions obtained to the combined phase and amplitude nulling problem (limited only by computer accuracy). Nevertheless quite acceptable results can be obtained in low sidelobe applications where the phase perturbations required to place nulls in the pattern are small. In Reference 12 we incorporated the solution of Reference 11 in an iterative method with which, in applications to patterns of low sidelobes, we were able to form nulls with a depth fully comparable to those obtained with the exact combined phase and amplitude solution. The iterative method failed to work well, however, when nulls were desired at closely spaced locations where the unperturbed pattern had high sidelobe values. In Reference 15 we investigated an approach to phase-only null synthesis consisting of starting with the exact solution to the combined phase and amplitude null synthesis problem and approximating the weight perturbations in a least-squares sense by phase-only weight perturbations. This was shown to be accomplished simply by setting the phase-only perturbations equal to the phase part of the combined phase and amplitude perturbations and ignoring the change in amplitude. The null depths achieved by this method were in general poor compared to those obtained with either the simple linearization method or the iterative linearization method, although some slight degree of nulling was obtained in situations where either or both of the other two methods failed to work at all; namely, nulling in patterns with high sidelobe levels, and/or many imposed null locations.

In this report we describe still another approach to the problem of phase-only null synthesis with minimized weight perturbations: that of nonlinear programming. A large amount of work has been devoted to analytic and numerical methods for solving the so-called nonlinear programming problem; that is, the problem of minimizing or maximizing a nonlinear objective function of several variables subject to

15. Shore, R.A. (1982) Phase-Only Nulling As a Least-Squares Approximation to Complex Weight Nulling, RADC-TR-82-129, AD A118722.

a set of nonlinear equality and/or inequality constraints in the variables. (The minimized weight perturbation phase-only null synthesis problem is an example of a nonlinear programming problem.) From the literature we have selected two particular numerical methods with readily available computer codes and applied them to the null synthesis problem. These methods give impressive results in general although convergence problems are encountered in some situations.

2. FORMULATION OF PHASE-ONLY NULL SYNTHESIS AS A NONLINEAR PROGRAMMING PROBLEM

We consider a linear array of equispaced isotropic elements (see Figure 1). The interelement spacing is d and the phase reference center is assumed to be the center of the array. Let w_n, $n = 1, 2, \ldots, N$, be the complex weight of the nth array element. Then the array field pattern, p(u) is

$$p(u) = \sum_{n=1}^{N} w_n e^{j d_n u}$$

where

$$d_n = \frac{N-1}{2} - (n-1), \quad n = 1, 2, \ldots, N$$

and

$$u = kd \sin \theta$$

with

$$k = \frac{2\pi}{\lambda}$$

and θ the angle measured from broadside to the array. The $\{d_n\}$ are odd-symmetric with respect to the phase reference center; that is,

$$d_n = -d_{N-n+1}, \quad n = 1, 2, \ldots, N.$$

Let a_n, $n = 1, 2, \ldots, N$, be a given taper, assumed to be symmetric with respect to the phase reference center, of the amplitudes of the element excitations, and let u_s be the direction of the peak of the array pattern corresponding to a linear phase shift of the element excitations. Then the array coefficients are

$$w_{on} = a_n e^{-j d_n u_s}, \quad n = 1, 2, \ldots, N .$$

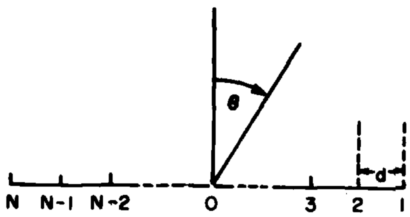

Figure 1. Geometry of Array

We wish to find the set of perturbations, ϕ_n, $n = 1, 2, \ldots, N$, of the phases of the element weights that will: (a) result in a perturbed pattern with nulls at a set of prescribed locations, u_k, $k = 1, 2, \ldots, M$, and (b) be as small as possible in the sense of minimizing either the sum of the squares of the absolute values of the total weight perturbations

$$\sum_{n=1}^{N} \left(a_n \left| e^{j\phi_n} - 1\right|\right)^2 = 4 \sum_{n=1}^{N} \left[a_n \sin\left(\frac{\phi_n}{2}\right)\right]^2 \tag{1}$$

or the sum of the squares of the absolute values of the weight perturbations relative to the amplitudes of the weights,

$$\sum_{n=1}^{N} \left| e^{j\phi_n} - 1\right|^2 = 4 \sum_{n=1}^{N} \sin^2\left(\frac{\phi_n}{2}\right) . \tag{2}$$

Since the perturbed coefficients are

$$w_n = a_n e^{-j d_n u_s} e^{j\phi_n}$$

the equations constraining the perturbed pattern to have nulls at the locations $u = u_k$, $k = 1, 2, \ldots, M$, are

$$\sum_{n=1}^{N} a_n e^{j[d_n (u_k - u_s) + \phi_n]} = 0, \quad k = 1, 2, \ldots, M . \tag{3}$$

Separating the real and imaginary parts of Eq. (3) we obtain

$$\sum_{n=1}^{N} a_n \cos[d_n(u_k - u_s) + \phi_n] = 0 \tag{4a}$$

$$\sum_{n=1}^{N} a_n \sin[d_n(u_k - u_s) + \phi_n] = 0 \, . \tag{4b}$$

The problem can then be stated as follows: find the set of phase perturbations, ϕ_n, $n = 1, 2, \ldots, N$, that minimizes either

$$4 \sum_{n=1}^{N} \left[a_n \sin\left(\frac{\phi_n}{2}\right)\right]^2 \tag{5a}$$

or

$$4 \sum_{n=1}^{N} \sin^2 \left(\frac{\phi_n}{2}\right) \tag{5b}$$

and that satisfies the null constraints Eq. (4). A further set of inequality constraint can be added if desired; namely,

$$-\pi < \phi_n \leq \pi, \; 1, 2, \ldots, N \, . \tag{6}$$

The problem, so expressed, is an example of a nonlinear programming problem, a problem in which a nonlinear objective function of several variables is to be minimized (or maximized) subject to a set of, in general, nonlinear equality and/or inequality constraints. Here the objective function, Eq. (5a) or Eq. (5b), and the equality constraints Eq. (4), are nonlinear. The inequality constraints (6) are linear.

Much work has been, and continues to be, devoted to developing efficient numerical algorithms and computer codes for solving nonlinear programming problems. Good recent summaries of the subject are to be found in References 16-18. From this literature we have selected two computer codes that are readily available, LPNLP and VMCON, for use on the minimized phase-only perturbation nulling problem.

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16. Sandgren, E. (1977) The Utility of Nonlinear Programming Algorithms, Ph.D. Thesis, Purdue University, December 1977.
17. Schittkowski, K. (1980) Nonlinear Programming Codes: Information, Tests, Performance, Springer-Verlag, New York.
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The code LPNLP is described at length in Reference 19. It utilizes an augmented Lagrangian method. The code VMCON is described in Reference 20 and utilizes an iterative quadratic programming method to solve the nonlinear programming problem. The results obtained with these codes are described in the next section.

3. RESULTS AND DISCUSSION

To examine the performance of LPNLP and VMCON, the algorithms were first tested on the problem of imposing nulls in the pattern of a 41 element array with uniform amplitude and half wavelength interelement spacing. The amplitudes of the elements were set equal to unity. A series of sets of imposed null locations was used starting with one null at 4.0°, then two nulls at 4.0° and 4.6°, up through five nulls at 4.0°, 4.6°, 5.2°, 5.8°, and 6.4°. In all cases the number of unknown phase perturbations was reduced by a factor of a half to twenty by assuming odd-symmetry of the phases with respect to the reference center, an assumption equivalent to that of assuming the perturbed pattern to be real.* The constraints Eq. (4b) are then automatically satisfied.

Both LPNLP and VMCON were run in double precision on a CDC 6600 computer. In running LPNLP the self-scaling mode (ISS = 1) was used with no automatic reset to the gradient direction (IRESET = 0). The convergence parameters were set at $\epsilon_1 = 1.0 \times 10^{-10}$, $\epsilon_2 = \epsilon_3 = 1.0 \times 10^{-8}$. In running VMCON, the tolerance was set at 1.0×10^{-10}. Both programs were run with the unknown phase perturbations set initially to zero.

In Table 1 we show null depths and computation time for the test problem. It is seen that for up to three nulls, LPNLP was the more efficient algorithm, while for four nulls VMCON gave the superior performance. For the five null case,

19. Pierre, D.A., and Lowe, M.J. (1975) Mathematical Programming Via Augmented Lagrangians: An Introduction With Computer Programs, Addison-Wesley, Reading, Massachusetts.

20. Crane, R.L. et al (1980) Solution of the General Nonlinear Programming Problem With Subroutine VMCON, Argonne National Laboratory Report ANL-80-64.

*This topic is the subject of a separate report[21] in which it is proven that for one imposed null the minimum phase perturbations always have odd-symmetry, and in which additional evidence is presented for supposing that in general the minimum phase-only perturbations required to impose nulls at a set of specified pattern locations are odd-symmetric. It was found, for example, that LPNLP and VMCON always arrived at an odd-symmetric set of phase perturbations even if the assumption of odd-symmetry was not made at the outset.

21. Shore, R.A. (1983) On the Odd-Symmetry of Minimum Phase-Only Perturbations, RADC-TR-83-26.

LPNLP converged extremely slowly, failing to meet the desired tolerance in more than 1000 sec CP time, while VMCON terminated without convergence. When the algorithms both converged they did so to the same solution to within a very small ($< 10^{-6}$) difference attributable to the fact that the two programs use different criteria for termination.

Table 1. Comparison of Performance of LPNLP and VMCON on Phase-Only Nulling in the Pattern of a 41 Element, Uniform Amplitude Array With $\lambda/2$ Spacing. Nulls imposed at the series of locations 4.0, 4.6, 5.2, 5.8 and 6.4 degrees

Number of Nulls	1		2		3		4		5	
	Null Depth (dB)	CP Time (sec)	Null Depth (dB)	CP Time (sec)	Null Depth (dB)	CP Time (sec)	Null Depth (dB)	CP Time (sec)	Null Depth (DB)	CP Time (sec)
LPNLP	-255	2.9	<-289	5.7	<-265	12.0	<-249	68.9	<-186	1007
VMCON	-266	25.4	<-254	32.5	<-275	39.5	<-283	34.4	----	----

Some measure of the numerical difficulty of phase-only nulling is the size of the phase perturbations required to impose nulls. It is possible to interpret the phase-only nulling solution as the superposition of the original (unperturbed) pattern and a set of cancellation beams, one for each imposed null location.[1, 11] For null locations spaced relatively far apart there is little interference between the cancellation beams, and the resulting phase perturbations are small. As the null locations are moved closer together, the beams increasingly interfere with each other, and the required phase perturbations increase correspondingly. The magnitude of the phase perturbations is also dependent on the sidelobe level of the original pattern at the desired null locations. High sidelobes require a larger change in the pattern for nulling than do low sidelobes and consequently are associated with larger phase perturbations than are low sidelobes. For small phase perturbations, the phase-only null synthesis problem approaches linearity[1, 11-13] and so is much easier to solve numerically than is the case when large phase perturbations are required and the nulling problem is essentially nonlinear. In this context, the above test problem was intentionally chosen to be a severe test of the nonlinear programming algorithms, requiring nulls to be imposed at closely spaced locations in a near-in sidelobe region of a uniform array. In Table 2 we have tabulated the average of the absolute values of the phase perturbations and the value of the objective function

$$4 \sum_{n=1}^{N} \sin^2\left(\frac{\phi_n}{2}\right)$$

as a function of the number of imposed nulls. The average absolute phase perturbation increases by a factor of three and the objective function by an order of magnitude from the one-null to the five-null case. Note the especially sharp increase for the four- and five-null cases.

Table 2. Average Absolute Phase Perturbation and the Objective Function $4\sum_n \sin^2(\phi_n/2)$ for Phase-Only Nulling in the Pattern of a 41 Element Uniform Amplitude Array With $\lambda/2$ Spacing. Nulls imposed at the series of locations 4.0, 4.6, 5.2, 5.8, and 6.4 degrees using LPNLP and VMCON

Number of Nulls	1	2	3	4	5
Average Absolute Phase Perturbation (radians)	0.2838	0.3073	0.3321	0.5365	0.8803
Objective Function	3.7887	4.5743	7.5406	19.993	36.665

It is of interest to compare the performance of LPNLP and VMCON with that of the iterative linearization method discussed in Reference 12. In Table 3 we give computation time, null depth, average absolute phase perturbation, and the value of the objective function for the same test problem, for one, two, and three nulls. Comparing Table 3 with Tables 1 and 2, it is seen that the iterative linearization method performs almost as well as the nonlinear programming methods as regards minimizing the objective function, and is considerably more efficient as regards CP time. For four nulls (at 4.0°, 4.6°, 5.2°, and 5.8°) however, the iterative linearization method failed to converge. In phase-only null synthesis problems requiring large phase perturbations, the iterative linearization method is not effective because the procedure is based on the linearization approximation $e^{j\phi_n} \approx 1 + j\phi_n$ which is good only for small ϕ_n.

To underscore the point made above about the test problem being a difficult one, in Tables 4 and 5 we have tabulated the same quantities as in Tables 1 and 2 respectively for a series of nulls imposed in the pattern of a 41 element array with uniform amplitude and half-wavelength spacing at the locations 25, 35, 45, 55, and 65 degrees. Note the marked decrease in computation time, especially for the cases with the higher number of nulls, and the much lower average absolute phase perturbation and objective function for this series of null locations as compared with the former set. The effect on the pattern of the much larger phase perturbations of the first test problem as compared with the second can be seen in Figures 2 and 3 where we show the original and perturbed patterns for the four-null case of the first and second test problems respectively. Note how much more closely the perturbed pattern follows the original pattern in Figure 3 than it does for the first test problem in Figure 2 where the large phase perturbations result in a marked change over most of the range of the pattern. The relatively small phase perturbations required in the second test problem make it possible for the iterative linearization method to be applied here to even the five-null case. The results for the series of nulls are shown in Table 6.

As one further test of the performance of LPNLP and VMCON on phase-only null synthesis problems, the two programs were run on the problem of imposing nulls at the four locations 14.70°, 15.28°, 15.86°, and 16.44° in the pattern of a 41 element array with half-wavelength spacing, with a 20 dB Chebyshev taper. The iterative linearization method had been tried without success on this problem as described in Reference 12. VMCON solved both the nulling problem with minimization of absolute weight perturbations and the null synthesis problem with minimization of relative weight perturbations [see Eqs. (1, 2)] in about 40 sec CP time each with null depths < -250 dB, but LPNLP failed to converge to a solution in either problem. Further work is needed to understand clearly the reasons for this failure and for the difficulties encountered in the five-null case of the first test problem. In general, however, it can be said that either LPNLP or VMCON provides an effective method for calculating the phases required for minimized weight perturbation, phase-only, null synthesis.

Table 3. Performance of the Iterative Linearization Method on Phase-Only Nulling in the Pattern of a 41 Element, Uniform Amplitude Array With $\lambda/2$ Spacing. Nulls imposed at the series of locations 4.0, 4.6, and 5.2 degrees

Number of Nulls	1	2	3
Null Depth (dB)	-298	<-271	<-275
CP Time (sec)	1.1	1.4	2.1
Average Absolute Phase Perturbation (radians)	0.2846	0.3088	0.3424
Objective Function $4 \sum_n \sin^2 (\phi_n/2)$	3.7789	4.5750	7.5594

Table 4. Performance of LPNLP and VMCON on Phase-Only Nulling in the Pattern of a 41 Element, Uniform Amplitude Array With $\lambda/2$ Spacing. Nulls imposed at the series of locations 25, 35, 45, 55, and 65 degrees

Number of Nulls	1		2		3		4		5	
	Null Depth (dB)	CP Time (sec)	Null Depth (dB)	CP Time (sec)	Null Depth (dB)	CP Time (sec)	Null Depth (dB)	CP Time (sec)	Null Depth (dB)	CP Time (sec)
LPNLP	-276	2.3	<-275	3.6	<-281	4.7	<-277	5.8	<-282	7.1
VMCON	-283	12.1	<-280	12.1	<-239	15.0	<-240	14.5	<-241	16.0

Table 5. Average Absolute Phase Perturbation and the Objective Function $4\sum_n \sin^2(\phi_n/2)$ for Phase-Only Nulling in the Pattern of a 41 Element Uniform Amplitude Array With $\lambda/2$ Spacing. Nulls imposed at the series of locations 25, 35, 45, 55, and 65 degrees using LPNLP and VMCON

Number of Nulls	1	2	3	4	5
Average Absolute Phase Perturbation (radians)	0.04431	0.04415	0.05320	0.05405	0.06275
Objective Function	0.09523	0.1261	0.1906	0.1972	0.2417

Table 6. Performance of the Iterative Linearization Method on Phase-Only Nulling in the Pattern of a 41 Element, Uniform Amplitude Array With $\lambda/2$ Spacing. Nulls imposed at the series of locations 25, 35, 45, 55, and 65 degrees

Number of Nulls	1	2	3	4	5
Null Depth (dB)	-288	<-283	<-275	<-279	<-272
CP Time (sec)	0.5	0.5	0.7	0.7	0.8
Average Absolute Phase Perturbation (radians)	0.04431	0.04416	0.05324	0.05408	0.06280
Objective Function $4\sum_n \sin^2(\phi_n/2)$	0.09523	0.1261	0.1907	0.1972	0.2417

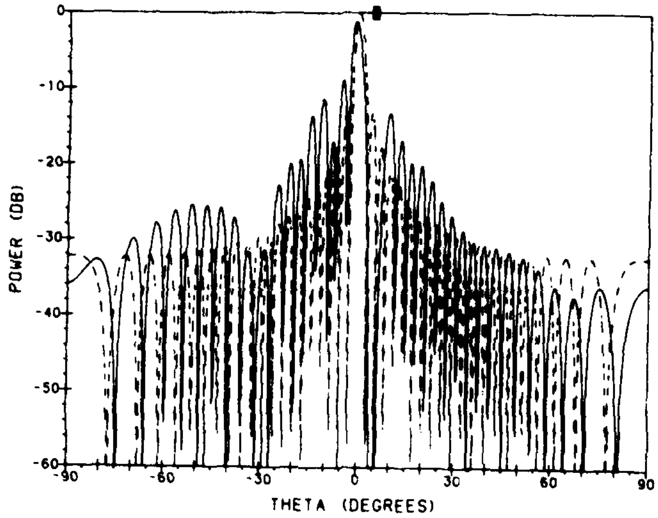

Figure 2. Original (----) and Perturbed (——) Pattern for Nulls Imposed at 4.0°, 4.6°, 5.2°, and 5.8° in the Pattern of a 41-Element Uniform Amplitude Array With λ/2 Spacing

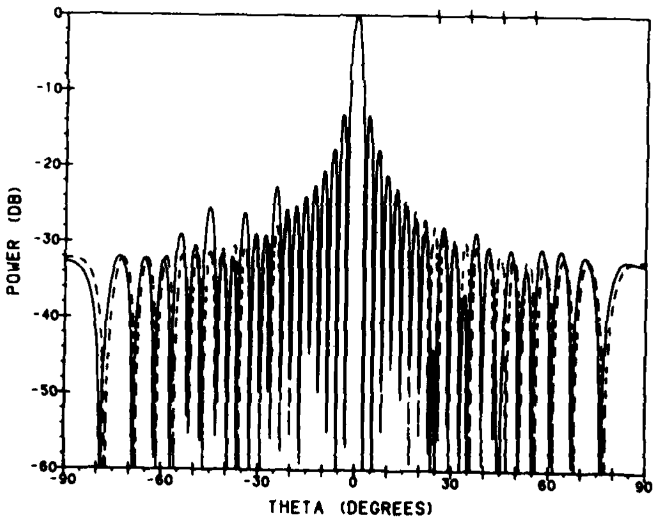

Figure 3. Original (----) and Perturbed (——) Pattern for Nulls Imposed at 25°, 35°, 45°, and 55° in the Pattern of a 41-Element Uniform Amplitude Array With λ/2 Spacing

4. CONCLUSIONS

In this report we have described the performance of two nonlinear programming computer codes, LPNLP and VMCON, on the problem of calculating the minimum phase-only weight perturbations required to impose nulls at specified locations in a linear array antenna pattern. Both codes are effective in general in solving the phase-only null synthesis problem, but convergence problems may be encountered if the nulling requirements become too severe: for example, multiple, closely spaced nulls in high sidelobe regions of the pattern.

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